

QUADRISECANT APPROXIMATION OF HEXAGONAL TREFOIL KNOT

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ABSTRACT. It is known that every nontrivial knot has at least two quadrisecants. Given a knot, we mark each intersection point of each of its quadrisecants. Replacing each subarc between two nearby marked points with a straight line segment joining them, we obtain a polygonal closed curve which we will call the quadrisecant approximation of the given knot. We show that for any hexagonal trefoil knot, there are only three quadrisecants, and the resulting quadrisecant approximation has the same knot type.

1. PRELIMINARIES

A *knot* is a locally flat simple closed curve in \mathbb{R}^3 . Two knots said to be equivalent if there is an orientation preserving homeomorphism of \mathbb{R}^3 onto \mathbb{R}^3 carrying one to the other. The equivalence class of a knot under this equivalence relation is called its *knot type*. A knot is said to be *nontrivial*, if it does not have the knot type of a planar circle.

A *quadrisecant* of a knot K is a straight line L such that $K \cap L$ has at least four components [2, 6].

Theorem 1 (Pannwitz). *Every nontrivial knot has at least two quadrisecants.*

A *polygonal knot* is a knot which is the union of finitely many straight line segments. Each maximal line segment of a polygonal knot is called an *edge* and its end points are called *vertices*.

Theorem 2 (Jin-Kim). *The trefoil knot[†] can be constructed as a polygonal knot with at least six edges.*

2. QUADRISECANTS OF A HEXAGONAL TREFOIL KNOT

A polygonal knot is said to be in *general position* if no three vertices are collinear and no four vertices are coplanar. It is clear that a quadrisecant of a polygonal knot in general position intersects the knot in finitely many points.

Let K be a polygonal knot. A triangular disk Δ determined by a pair of adjacent edges of K is said to be *reducible* if the Δ intersects K only in the two edges, and *irreducible* otherwise. If K is a polygonal knot in general position with the least number of edges in its knot type, then every triangular disk determined by a pair of adjacent edges of K is irreducible.

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[†]3₁ in [7]

Let K denote a hexagonal knot with vertices v_1, \dots, v_6 , and edges $e_{i,i+1}$ joining v_i and v_{i+1} , for $i = 1, \dots, 6$, where the subscripts are written modulo 6. For $i = 1, \dots, 6$, the triangular disk with vertices at v_{i-1}, v_i, v_{i+1} will be denoted as Δ_i . In [3], Huh and Jeon showed that in a hexagonal trefoil knot, the edges and the triangular disks intersect in a special pattern as follows:

Proposition 3 (Huh-Jeon). *If K is a hexagonal trefoil knot, then the triangular disks $\Delta_1, \dots, \Delta_6$ are irreducible. Furthermore, up to a cyclic relabeling of the vertices, the following are the only nonempty intersections among edges and triangular disks:*

$$\begin{aligned} \text{int } e_{23} \cap \text{int } \Delta_5, & \quad \text{int } e_{23} \cap \text{int } \Delta_6, \\ \text{int } e_{45} \cap \text{int } \Delta_1, & \quad \text{int } e_{45} \cap \text{int } \Delta_2, \\ \text{int } e_{61} \cap \text{int } \Delta_3, & \quad \text{int } e_{61} \cap \text{int } \Delta_4. \end{aligned}$$

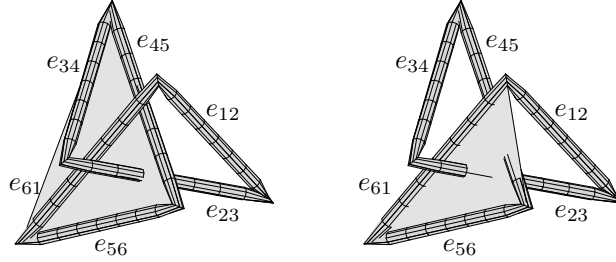


FIGURE 1. $e_{23} \cap \Delta_5$ and $e_{23} \cap \Delta_6$

Theorem 4. *Every hexagonal trefoil knot has exactly three quadrisecants.*

Proof. We first show that no three consecutive edges of K are coplanar. Suppose that three consecutive edges of K , say e_{12}, e_{23} and e_{34} , lie on a plane P . Then all vertices other than v_5 and v_6 lie on P . Therefore the height function of K in a normal direction to P has only one local maximum point and one local minimum point. Since such a knot is trivial, it contradicts that K is a trefoil knot.

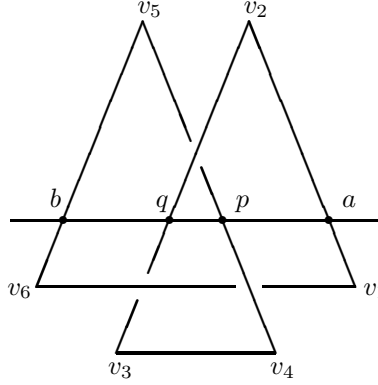
If L is a quadrisecant of K , then there are four distinct edges of K corresponding to four points of $K \cap L$. If any three of these edges are consecutive along K , then they are coplanar. Therefore, by the above argument, there are only three possible sets of four edges meeting L :

$$\{e_{12}, e_{23}, e_{45}, e_{56}\}, \{e_{23}, e_{34}, e_{56}, e_{61}\}, \{e_{34}, e_{45}, e_{61}, e_{12}\}.$$

We show that there exists exactly one quadrisecant in each of the above three cases. We may assume that the vertices of K are labeled so that the edges and the triangular disks of K intersect as stated in Proposition 3.

By cyclically relabeling the vertices of K , we only need to show that there exists exactly one quadrisecant meeting the edges $\{e_{12}, e_{23}, e_{45}, e_{56}\}$. Let P_2 and P_5 be the planes containing the triangular disks Δ_2 and Δ_5 , respectively. We show that $P_2 \cap P_5$ is the quadrisecant we are seeking.

Notice that $e_{45} \cap P_2 \neq \emptyset$ and $e_{23} \cap P_5 \neq \emptyset$. Let $p = e_{45} \cap P_2$ and $q = e_{23} \cap P_5$. Then $P_2 \cap P_5$ is the line through the two points p and q . Notice that the endpoints of e_{23} lie on the opposite sides of P_5 . Let P_5^+ and P_5^- be the open half spaces

FIGURE 2. $p, q, a, b \in P_2 \cap P_5$

divided by P_5 containing v_2 and v_3 , respectively. Since $v_3 \in P_5^-$ and $v_4, v_5 \in P_5$, we see that Δ_4 lies in P_5^- except along e_{45} . Since e_{61} and Δ_4 intersect in their interiors, we also know that e_{61} lies in P_5^- except at v_6 , hence $v_1 \in P_5^-$. Then, we see that e_{12} intersects P_5 in its interior. Let $a = e_{12} \cap P_5$. Then $a \in P_2 \cap P_5$.

Notice that the endpoints of e_{45} lie on the opposite sides of P_2 . Let P_2^+ and P_2^- be the open half spaces divided by P_2 containing v_5 and v_4 , respectively. Similarly as above, Δ_1 lies in P_2^- except along e_{12} . Since e_{61} and Δ_3 intersect in their interiors, we also know that e_{61} lies in P_2^- except at v_1 , hence $v_6 \in P_2^-$. Therefore we have another point $b = e_{56} \cap P_2$ in $P_2 \cap P_5$. This completes the proof. \square

3. QUADRISECANT APPROXIMATION

Let K be a knot which has finitely many quadrisecants intersecting K in finitely many points. The intersection points cut K into finitely many subarcs. Straightening each subarc with its endpoints fixed, one obtains a polygonal closed curve which we will call the *quadrisecant approximation* of K , denoted by \hat{K} .

Experiments on some knots with small crossings indicate that it might be true that \hat{K} is actually a knot having the knot type of K [5]. We show that the quadrisecant approximation \hat{K} of a hexagonal trefoil knot K is a trefoil knot of the same type and that the quadrisecants of \hat{K} are those three of K constructed in the proof of Theorem 4.

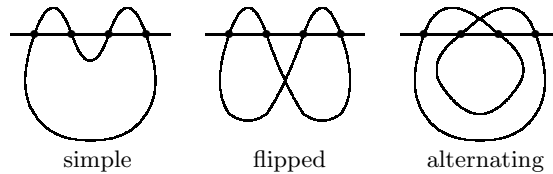


FIGURE 3. Three types of quadrisecants

For any knot, a quadrisecant is one of the three types, *simple*, *flipped* or *alternating*, according to the orders of the four intersection points along the quadrisecant and along the knot as indicated in Figure 3 [1]. It is easily seen from the proof of Theorem 4 and Figure 2 that the following lemma holds.

Lemma 5. *All quadrisecants of a hexagonal trefoil knot are alternating.*

Theorem 6. *If K is a hexagonal trefoil knot, then its quadrisecant approximation \hat{K} is also a trefoil knot that has the same knot type as K .*

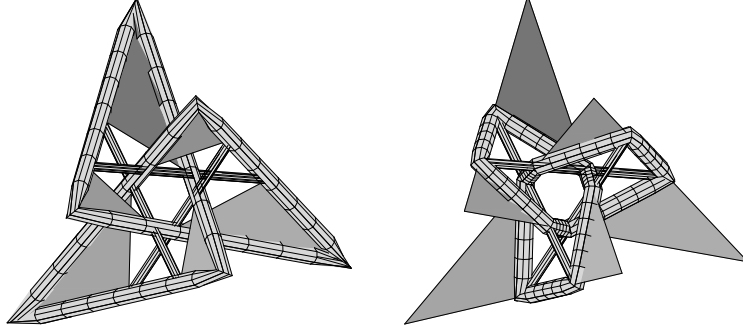


FIGURE 4. Quadrisecants and quadrisecant approximation of a hexagonal trefoil knot

Proof. Let K be a hexagonal trefoil knot whose vertices and edges are labeled so that it has the intersection pattern as described in Proposition 3. Let L_1, L_2, L_3 be the quadrisecants of K corresponding to the set of edges

$$\{e_{12}, e_{23}, e_{45}, e_{56}\}, \{e_{23}, e_{34}, e_{56}, e_{61}\}, \{e_{34}, e_{45}, e_{61}, e_{12}\},$$

respectively. Let p_{ij}^k denote the point $L_k \cap e_{ij}$ and let s_i^k denote the subarc of K which is the union of the segment of e_{i+1} from v_i to p_{i+1}^k and the segment of e_{i-1} between v_i and p_{i-1}^k .

We first observe the quadrisecant $L_1 = P_2 \cap P_5$ and the edges $e_{12}, e_{23}, e_{45}, e_{56}$. Notice that $K \cap L_1 = \{p_{12}^1, p_{23}^1, p_{45}^1, p_{56}^1\}$. We consider s_2^1 . If there is no p_{ij}^k 's in the interior of s_2^1 , the quadrisecant approximation \hat{K} has a self-intersection at p_{45}^1 . So, we need to show that there exists one of the p_{ij}^k 's in the interior of s_2^1 . Notice that e_{34} lies in P_5^- except at v_4 . Thus, p_{34}^3 , the intersection point of L_3 and e_{34} , lies in P_5^- . And note that p_{45}^3 lies on the plane P_5 . By Lemma 5, L_3 is an alternating quadrisecant. That is, along L_3 , the order of the p_{ij}^3 's is $p_{12}^3 p_{45}^3 p_{61}^3 p_{34}^3$. Thus, we know that p_{12}^3 lies in P_5^+ . So, we know that p_{12}^3 lies on the intersection of e_{12} and the interior of s_2^1 .

Now, we need to show that there are no intersection points of interior of s_2^1 and quadrisecants of K except at p_{12}^3 . Note that each of e_{ij} 's has exactly two p_{ij}^k 's. So, it is not hard to see that p_{23}^2 is the only candidate for any additional intersection of s_2^1 and quadrisecants of K . We show that p_{23}^2 doesn't lie on s_2^1 .

By Lemma 5, along L_2 , the order of p_{ij}^2 's is $p_{56}^2 p_{23}^2 p_{61}^2 p_{34}^2$. Let $\overline{L_2}$ be a segment which has the end points p_{56}^2 and p_{34}^2 . Since p_{34}^2 lies on e_{34} and e_{34} lies in P_5^- , p_{34}^2 lies in P_5^- . And note that p_{56}^2 lies on P_5 . Thus, p_{23}^2 , the point which lies between p_{56}^2 and p_{34}^2 , lies in P_5^- . So, we know that p_{23}^2 does not lie on s_2^1 .

Now, by a similar way, we can show that $p_{12}^1, p_{34}^3, p_{34}^2, p_{56}^2, p_{56}^1$ are the only points which lie on the interior of $s_1^3, s_3^3, s_4^3, s_5^1, s_6^2$, respectively.

Finally, let t_i^k be the line segment joining the end points of s_i^k . Let δ_{ik} be the triangle which is bounded by s_i^k and t_i^k . Since $\text{int } \Delta_a$ intersects only one edge of K transversely and the edge meets t_i^k , the interior of δ_{ik} does not meet K .

Let δ_i be the triangle with vertices v_i , p_{i-1}^k and p_{i+1}^l where k, l are determined by the property:

The segments $\overline{v_i p_{i-1}^k}$ and $\overline{v_i p_{i+1}^l}$ do not contain any quadriseccant point in their interior.

Note that $\delta_i \subset \delta_{ik}$. Thus δ_i does not meet K .

We consider the case of δ_2 . Note that Δ_2 does not meet $\text{int } \Delta_1$, $\text{int } \Delta_3$. Thus, we just observe whether each of $\delta_4, \delta_5, \delta_6$ meets δ_2 or not. Note that p_{45}^3, v_4 and p_{34}^2 lie in P_2^- , p_{45}^1 lies on P_2 , v_5 and p_{56}^2 lie in P_2^+ . Since $p_{45}^1 \notin \delta_2$, we have $(\delta_4 \cup \delta_5) \cap \delta_2 = \emptyset$.

Since $\text{int } \Delta_3$ lies on P_2^- and e_{61} intersects $\text{int } \Delta_3$ transversely, we know that p_{61}^2 and v_6 lie in P_2^- . And we also know that p_{56}^1 lies on P_2 but $p_{56}^1 \notin \delta_2$. So, δ_6 does not meet δ_2 . Thus, we conclude that δ_2 does not meet δ_i ($i \neq 2$). By the similar way, we say that if $i \neq j$, δ_i does not meet δ_j . So, we conclude that the quadriseccant approximation \hat{K} of K has the same knot type of K . Thus, \hat{K} is a trefoil knot. \square

Theorem 7. *If K is a hexagonal trefoil knot, then the quadriseccants of the quadriseccant approximation \hat{K} are just the three quadriseccants of K .*

The proof of Theorem 7 is a combination of the lemmas and corollaries that follow. Recall that P_i is the plane determined by the vertices v_{i-1}, v_i and v_{i+1} . Let O_i denote the edge of \hat{K} which is contained in e_{i+1} , and let N_i denote the edge of \hat{K} joining O_{i-1} and O_i . \hat{K} has twelve vertices $v_{i-1} = O_{i-1} \cap N_i$ and $v_{i+1} = N_i \cap O_i$.

Counting the number of adjacent pairs of edges among four edges meeting a quadriseccant in four distinct points, there are three types of quadriseccants.

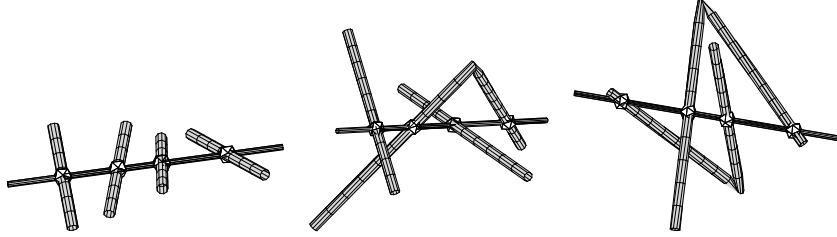


FIGURE 5. Quadriseccants of type-0, type-1 and type-2

If a quadriseccant intersects a polygonal knot at one or more vertices, then it can have more than one types.

Lemma 8. *If there exists a new quadriseccant of type-1 or type-2 of \hat{K} , then it lies on at least one of the P_i 's.*

Proof. The two adjacent edges of \hat{K} are either $O_{i-1} \cup N_i$ or $N_i \cup O_i$. As they are contained in P_i , the lemma holds. \square

Lemma 9. *For any P_i , there is no new quadriseccant lying on it.*

Proof. First, there is no line meeting O_{i-1} , N_i and O_i except at v_{i+1} and v_{i-1} . Next, note that there is no new quadriseccant meeting two points among the four points lying on the same quadriseccant L_k of K . \square

By the two previous lemmas, we have the following corollaries.

Corollary 10. *There is no new quadrisecant of type-1 or type-2 for \hat{K} .*

Corollary 11. *There is no new quadrisecant of type-0 meeting O_{i-1} and O_i .*

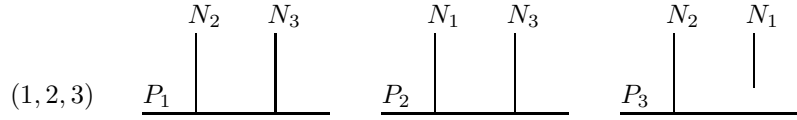
Lemma 12. *There is no new quadrisecant of type-0 for \hat{K} .*

Proof. By Corollary 11, we know that there is no quadrisecant of type-0 meeting O_{i-1} and O_i . Thus, if there exists a quadrisecant of type-0 meeting $O_i - O_j - O_k$, then it must be $O_1 - O_3 - O_5$ or $O_2 - O_4 - O_6$. Note that O_1 meets N_6 and N_1 , O_3 meets N_2 and N_3 and O_5 meets N_4 and N_5 . So, there is no quadrisecant of type-0 meeting $O_1 - O_3 - O_5$. Similarly, there is no quadrisecant of type-0 meeting $O_2 - O_4 - O_6$. Hence, there is no quadrisecant of type-0 meeting $O_i - O_j - O_k$.

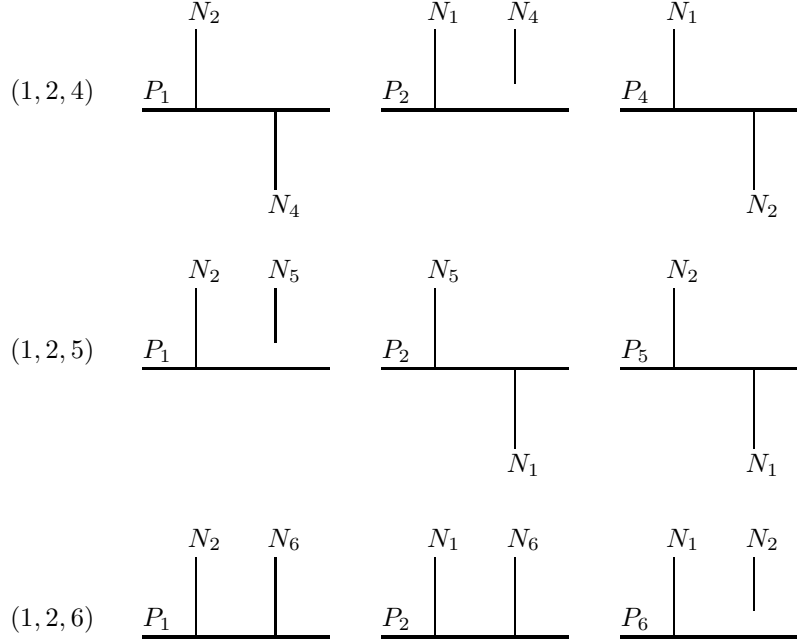
We only need to check the cases meeting $N_i - N_j - N_k$ or $N_i - N_j - O_k - O_l$.

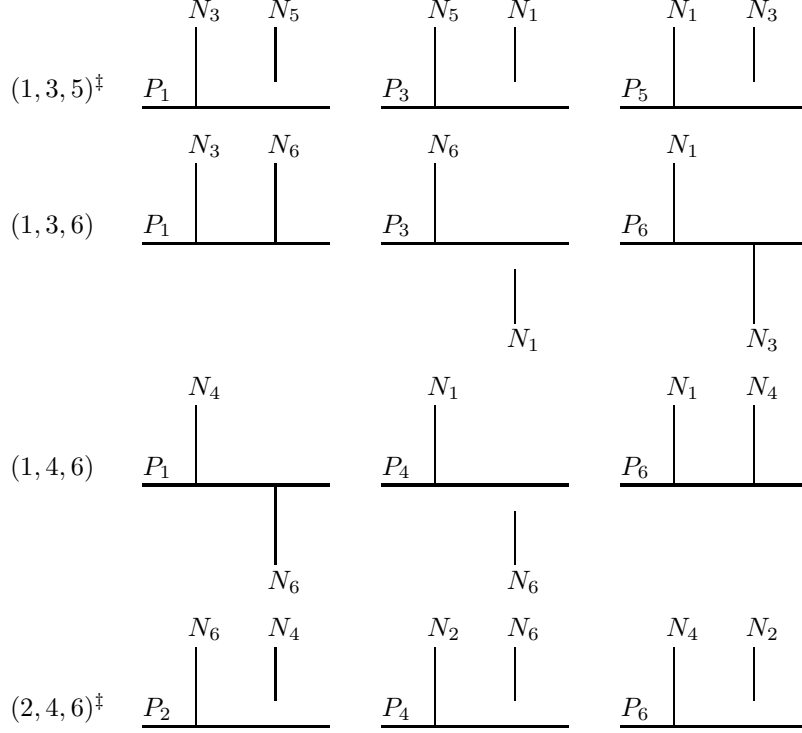
CASE 1. $N_i - N_j - N_k$ is not possible for any pairwise distinct triple $\{i, j, k\}$.

The following diagram indicates the case $\{i, j, k\} = \{1, 2, 3\}$ (and its order 3 cyclic relabelings $\{3, 4, 5\}$ and $\{5, 6, 1\}$). The edges N_j and N_k are on the same side of P_i . Except in the case of N_1 and P_3 , the edges N_j, N_k have one endpoint on P_i . Notice that N_i is contained in P_i . One can verify that there is no proper ordering of the intersections of the edges N_i, N_j, N_k along a line satisfying all the three parts of the diagram.



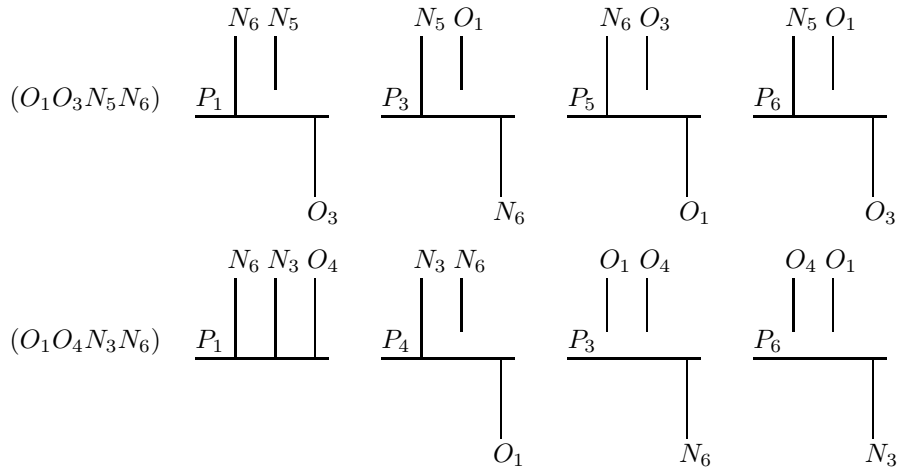
For the other cases (and their order 3 cyclic relabelings), we only show diagrams.



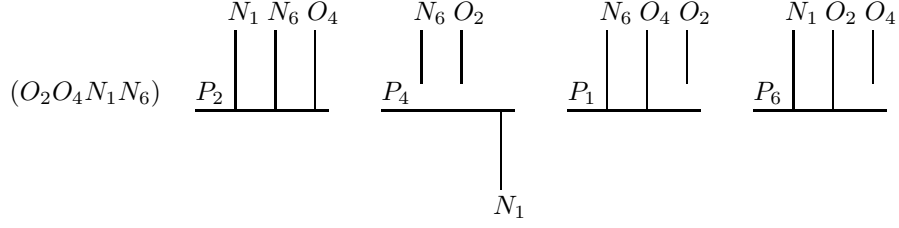


CASE 2. $N_i - N_j - O_k - O_l$ is not possible for any i, j, k and l with $i \neq j, k \neq l, l \neq \pm 1$.

In each of the following diagrams, we also consider order 3 cyclic relabelings and reverse cyclic relabelings. One can verify that there is no proper ordering of the intersections of the edges N_i, N_j, O_k, O_l along a line satisfying all the four parts of each of the diagrams below.



[‡]Stable under order 3 relabeling



This completes the proof. □

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